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Huang + Litzenberger

CHAPTER 6
VALUATION OF COMPLEX SECURITIES AND OPTIONS WITH PREFERENCE RESTRICTIONS

6.1. In Chapter 4, we made assumptions on the return distributions in order to derive linear valuation relations. In this chapter, we will first discuss valuation principles for complex securities in the framework of Chapter 5 without special assumptions on either the return distributions or individuals' utility functions. We will then derive explicit valuation expressions for risky assets under preference and distribution restrictions. In particular, the price of a European call option written on a stock when individuals' utility functions exhibit constant relative risk aversion and the option's underlying asset has a payoff structure that is jointly lognormally distributed with the aggregate consumption is explicitly computed. We also apply the pricing formula for a European call option to study the pricing of risky corporate debt. In the last section of this chapter, we derive a pricing relation similar to the CAPM for a particular class of risky assets.

The pricing relations derived in this chapter provide additional testable propositions concerning the pricing of complex securities such as common stocks and options. Some of these propositions will be empirically examined in Chapter 10.

6.2. Assume that individuals have homogeneous beliefs π_ω and utility functions that are time-additive and state-independent, denoted by u_{i0} and u_i, and assume that these utility functions are increasing, strictly concave, and differentiable. There are $N+1$ securities traded, indexed by $j = 0, 1, \ldots, N$. Individuals' time-0 endowments are units of time-0 consumption good and shares of traded securities. Security j is represented by its state dependent payoff structure $x_{j\omega}$. The 0-th security is a riskless discount bond with $x_{0\omega} = 1$ for all $\omega \in \Omega$.

We assume that the equilibrium allocation is Pareto optimal. We recall from Chapter 5 that under this condition, a representative agent with increasing and strictly concave utility functions u_0 and u_1 can be constructed, and the price of a primitive state contingent claim can be expressed as

$$\phi_\omega = \frac{\pi_\omega u_1'(C_\omega)}{u_0'(C_0)} \quad \forall \omega \in \Omega, \tag{6.2.1}$$

where ϕ_ω is the state price for state ω. A complex security may be viewed as a portfolio of elementary state contingent claims. Thus the price for security j is

$$S_j = \sum_{\omega \in \Omega} \phi_\omega x_{j\omega}. \tag{6.2.2}$$

Substituting (6.2.1) into (6.2.2) for ϕ_ω gives

$$S_j = E\left[\frac{u_1'(\tilde{C})}{u_0'(C_0)}\tilde{x}_j\right], \tag{6.2.3}$$

where we have used $\tilde{x}_j$ to denote the random time-1 payoff of security j. For the case of a riskless unit discount bond – a complex security that pays one unit of consumption at time 1 in all states, we have

$$S_0 = E\left[\frac{u_1'(\tilde{C})}{u_0'(C_0)}\right]. \tag{6.2.4}$$

As S_0 is the price of a unit discount bond, the riskless interest rate r_f is

$$r_f = \frac{1}{S_0} - 1. \tag{6.2.5}$$

By the strict monotonicity of the utility functions, $S_0 > 0$. This implies that $r_f > -1$. Substituting (6.2.5) into (6.2.4) gives

$$\frac{1}{1+r_f} = E\left[\frac{u_1'(\tilde{C})}{u_0'(C_0)}\right]. \tag{6.2.6}$$

Dividing both sides of (6.2.3) by S_j and using the definition of covariance, we can write

$$E[\tilde{r}_j - r_f] = -\left(E\left[\frac{u_1'(\tilde{C})}{u_0'(C_0)}\right]\right)^{-1} \operatorname{Cov}(\tilde{r}_j, u_1'(\tilde{C})/u_0'(C_0)), \tag{6.2.7}$$

where $\tilde{r}_j \equiv \tilde{x}_j/S_j - 1$ is the rate of return of security j. Substituting (6.2.6) into (6.2.7) gives an equilibrium relation for the risk premiums on securities:

$$E[\tilde{r}_j - r_f] = -(1+r_f)\operatorname{Cov}(\tilde{r}_j, u_1'(\tilde{C})/u_0'(C_0)). \tag{6.2.8}$$

By the fact that u_1 is strictly concave, the risk premium of a security is positive if and only if its random payoff at time 1 is positively correlated with the time-1 aggregate consumption. Note that in a two-period (period 0 and period 1) economy, by the strict monotonicity of utility functions, the time-1 aggregate consumption $\tilde{C}$ is equal to the time-1 aggregate endowment, which in turn is equal to time-1 aggregate wealth $\tilde{M}$. Therefore, (6.2.8) can be written as

$$E[\tilde{r}_j - r_f] = -(1+r_f)\operatorname{Cov}(\tilde{r}_j, u_1'(\tilde{M})/u_0'(C_0)). \tag{6.2.9}$$

That is, the risk premium of a security is positive if and only if its time-1 random payoff is positively correlated with the time-1 aggregate wealth. The intuition behind this result is the same as that of the Capital Asset Pricing Model. One unit of consumption in a state where the aggregate resource is abundant is less valuable than one unit of consumption in a state where the aggregate resource is scarce. Therefore, a security that pays more in states where the aggregate consumption/wealth is low is more valuable than a security that pays more in states where the aggregate consumption/wealth is high, *ceteris paribus*. As a result, the price for the former will be higher than that for the latter, and the rate of return on the former will be lower than that on the latter.

The market portfolio is a portfolio of traded securities. Thus its rate of return $\tilde{r}_m$ must also satisfy (6.2.9):

$$E[\tilde{r}_m - r_f] = -(1+r_f)\text{Cov}(\tilde{r}_m, u_1'(\tilde{M})/u_0'(C_0)). \tag{6.2.10}$$

Relation (6.2.10) implies that the risk premium on the market portfolio must be strictly positive, as $r_f > -1$ and u_1' is strictly decreasing so that $\text{Cov}(\tilde{r}_m, u_1'(\tilde{M}))$ is strictly negative. (Readers should compare this with the result of Section 4.14.) Substituting (6.2.10) into (6.2.9) gives

$$E[\tilde{r}_j - r_f] = \frac{\text{Cov}(\tilde{r}_j, u_1'(\tilde{M}))}{\text{Cov}(\tilde{r}_m, u_1'(\tilde{M}))}E[\tilde{r}_m - r_f]. \tag{6.2.11}$$

In equilibrium, the risk premium of security j is proportional to that of the market portfolio. The proportionality is equal to the ratio of the covariance of $\tilde{r}_j$ and $u_1'(\tilde{M})$ and the covariance of $\tilde{r}_m$ and $u_1'(\tilde{M})$.

6.3. We will now specialize the pricing relation of (6.2.11) by considering a class of utility functions. Assume that individuals' utility functions for time–1 consumption are power functions:

$$u_i(z) = \frac{1}{B-1}(A_i + Bz)^{1-\frac{1}{B}} \tag{6.3.1}$$

and that there is a riskless asset. Note that B is assumed to be constant across individuals. We also assume that the u_i's are increasing and strictly concave over the relevant region. Recall from Chapter 5 that the Pareto optimal sharing rules for time–1 consumption are linear in this case and can be attained if there is a riskless asset and if all assets are traded. Therefore, by the assumption that individuals are endowed only with holdings of shares of traded securities and time–0 consumption good in a securities markets economy, the equilibrium allocation is Pareto optimal. Moreover, Chapter 5 also shows that there exists a representative agent whose utility function for time–1 consumption is a power function with the same B:

$$u_1(z) = \frac{1}{B-1}(A + Bz)^{1-\frac{1}{B}}, \tag{6.3.2}$$

where $A \equiv \sum_{i=1}^{I} A_i$. Hence (6.2.11) becomes

$$E[\tilde{r}_j - r_f] = \frac{\text{Cov}(\tilde{r}_j, (A+B\tilde{M})^{-\frac{1}{B}})}{\text{Cov}(\tilde{r}_m, (A+B\tilde{M})^{-\frac{1}{B}})}E[\tilde{r}_m - r_f]. \tag{6.3.3}$$

Note that when $B = -1$, individuals' utility functions are quadratic and (6.3.3) becomes the familiar CAPM relation. When $B = -1/2$, the representative agent's utility function for time–1 consumption is a cubic function:

$$u_1(z) = -\frac{2}{3}(A - \frac{1}{2}z)^3.$$

The marginal utility in this case is

$$u_1'(z) = (A - \frac{1}{2}z)^2,$$

so u_1 is increasing and strictly concave for $z < 2A$. Thus if $\tilde{M} < 2A$, (6.3.3) becomes

$$\begin{aligned} E[\tilde{r}_j - r_f] &= \frac{\text{Cov}(\tilde{r}_j, (A - \tilde{M}/2)^2)}{\text{Cov}(\tilde{r}_m, (A - \tilde{M}/2)^2)}E[\tilde{r}_m - r_f] \\ &= \frac{A\text{Cov}(\tilde{r}_j, \tilde{r}_m)W_{m0} - \frac{1}{4}\text{Cov}(\tilde{r}_j, \tilde{r}_m^2)W_{m0}^2}{A\text{Var}(\tilde{r}_m)W_{m0} - \frac{1}{4}\text{Cov}(\tilde{r}_m, \tilde{r}_m^2)W_{m0}^2}E[\tilde{r}_m - r_f], \end{aligned} \tag{6.3.4}$$

where W_{m0} is the time–0 total value of traded securities. From (6.3.4) we notice that the risk premium of risky asset j depends not only upon the covariance of its return with the return on the market portfolio but also upon $\text{Cov}(\tilde{r}_j, \tilde{r}_m^2)$, which we term *coskewness*. The risk premium on asset j is higher the higher the covariance $\text{Cov}(\tilde{r}_j, r_m)$ and the lower the *coskewness* with the market portfolio. The latter of these two effects is a consequence of the fact that a cubic utility function exhibits preference towards the skewness of time–1 random consumption. The higher the coskewness of a security's return with that of the market portfolio, the more attractive it is to the individuals, *ceteris paribus*. Therefore, it will sell for a higher price and thus will have a lower expected rate of return.

6.4. Chapter 5 showed that call options help to achieve nonlinear sharing rules. Moreover, the prices of call options on aggregate consumption are sufficient to price *any* complex security in an allocationally efficient securities market where individuals' preferences are represented by time–additive, increasing, and strictly concave von Neumann–Morgenstern utility functions. In Sections 6.5 through 6.8, we will discuss certain properties of option prices that can be established by using purely arbitrage arguments. Section 6.9 gives some comparative statics of option prices as functions of their underlying asset prices and exercise prices. In later sections, an option pricing formula will be derived under the assumption that the payoff of the option's underlying asset and time–1 aggregate consumption are jointly lognormally distributed and that the representative agent's utility functions exhibit constant relative risk aversion.

6.5. Recall that a European call option is a security that gives its holder the right to purchase a share of its underlying security at a fixed exercise price on the maturity date of the option. Let $\tilde{x}_j(k)$ be the time–1 payoff from a European call on one share of the j–th security maturing at time 1 with an exercise price of k, and let $p_j(S_j, k)$ be the price of this call at time 0 when its underlying stock price is S_j. Since an option gives its holder the right but not the obligation to exercise on the maturity date,

$$\tilde{x}_j(k) = \begin{cases} \tilde{x}_j - k & \text{if } \tilde{x}_j - k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{x}_j$ is the random payoff of a share of the j–th security.

We claim that

$$p_j(S_j, k) \geq \max[S_j - k/(1 + r_f), 0], \qquad (6.5.1)$$

independent of individuals' utility functions and payoff distributions. The inequality will be strict if the probability that the option will be exercised is strictly between 0 and 1. We will prove now the strict inequality. Consider the following strategy: short sell one share of security j, buy one European call written on security j with an exercise price k maturing at time 1 and lend $k/(1+r_f)$ dollars at the riskless rate. This strategy has an initial cost equal to $p_j(S_j, k) - S_j + k/(1 + r_f)$ and has a time–1 payoff:

$$\begin{cases} \tilde{x}_j - k - \tilde{x}_j + k = 0 & \text{if } \tilde{x}_j \geq k, \\ -\tilde{x}_j + k > 0 & \text{if } \tilde{x}_j < k. \end{cases}$$

The time–1 payoff of this strategy is nonnegative and is strictly positive with a strictly positive probability, since there is a strictly positive probability that $\tilde{x}_j < k$. Therefore, its initial cost must be strictly positive to prevent something being created from nothing. That is, we must have

$$p_j(S_j, k) - S_j + k/(1 + r_f) > 0,$$

which is equivalent to

$$p_j(S_j, k) > S_j - k/(1 + r_f). \qquad (6.5.2)$$

Lastly, since a holder of the option only has the right and not the obligation to buy a share of its underlying security at the exercise price, the price of the option must be nonnegative. Moreover, by assumption there exists a strictly positive probability that the option will be exercised. Therefore, $p_j(S_j, k) > 0$. This observation together with (6.5.2) gives

$$p_j(S_j, k) > \max[S_j - k/(1 + r_f), 0].$$

which was to be shown. The intuition behind this inequality is as follows. The present value of an obligation to buy a share of security j at time 1 at a price k is $S_j - k/(1+r_f)$. When there exists a strictly positive probability that $\tilde{x}_j$ will be strictly less than k, the option *not* to buy has a strictly positive value. Thus the call option must be worth strictly more than $S_j - k/(1 + r_f)$. On the other hand, there exists a strictly positive probability that the option will be exercised. Hence, $p_j(S_j,k) > 0$.

6.6. In Section 5.19, by assuming that an option price is a twice differentiable function of its exercise price, we showed that an option price is a convex function of its exercise price. This property, as it turns out, holds more generally.

We want to show that

$$\alpha p_j(S_j,k) + (1-\alpha)p_j(S_j,\hat{k}) \geq p_j(S_j,\bar{k}), \tag{6.6.1}$$

where $\bar{k} = \alpha k + (1-\alpha)\hat{k}$ and $\alpha \in (0,1)$. Consider the strategy of buying α shares of the call option with an exercise price k and $(1-\alpha)$ shares of the call option with an exercise price $\hat{k}$, and short selling a share of call option with an exercise price $\bar{k}$. Without loss of generality, assume that $\hat{k} > k$. The time–1 payoff of this strategy is

$$\begin{cases} 0 & \text{if } \tilde{x}_j \leq k, \\ \alpha(\tilde{x}_j - k) > 0 & \text{if } k < \tilde{x}_j \leq \bar{k}, \\ (1-\alpha)(\hat{k} - \tilde{x}_j) > 0 & \text{if } \bar{k} < \tilde{x}_j \leq \hat{k}, \\ 0 & \text{if } \tilde{x}_j > \hat{k}, \end{cases}$$

which is nonnegative. Thus,

$$\alpha p_j(S_j,k) + (1-\alpha)p_j(S_j,\hat{k}) - p_j(S_j,\bar{k}) \geq 0,$$

which is just (6.6.1). When there is a strictly positive probability that $\tilde{x}_j \in (k,\hat{k}]$, the weak inequality becomes a strict inequality.

You are asked to prove in Exercise 6.2 that $p_j(S_j,k)$ is a decreasing function of k. Hence $p_j(S_j,k)$ is a decreasing and convex function of its exercise price.

In a frictionless market, buying an option on two shares of security j with an exercise $2k$ should be equal to buying two options with an exercise price k on security j. To see this, we simply note that the random payoffs of the former are identical to those of the latter.

6.7. An option on a positively weighted portfolio of securities with an exercise price k is less valuable than a positively weighted portfolio of options with equal exercise prices k. Consider a positively weighted portfolio of securities with weights α_j, $j = 1,\ldots,N$, where α_j denotes the portfolio weight on security j. Note that

$$\sum_{j=1}^{N} \alpha_j = 1 \quad \alpha_j \geq 0.$$

The time–0 cost and time–1 random payoff of this portfolio are

$$S^* \equiv \sum_{j=1}^{N} \alpha_j S_j$$

and

$$\tilde{x}^* \equiv \sum_{j=1}^{N} \alpha_j \tilde{x}_j,$$

respectively. Let $p^*(S^*,k)$ be the price of a European call written on the portfolio of securities with an exercise price k that expires at time 1. The time–1 random payoff of this option is

$$\max[\sum_{j=1}^{N} \alpha_j \tilde{x}_j - k, 0].$$

Since $\max[z,0]$ is a convex function of z, by the Jensen's inequality we have

$$\max[\sum_{j=1}^{N} \alpha_j \tilde{x}_j - k, 0] \leq \sum_{j=1}^{N} \alpha_j \max[\tilde{x}_j - k, 0].$$

Note that the right–hand side of the inequality is the time–1 random payoff of a portfolio of call options on individual securities with identical exercise prices k. Thus

$$p^*(S^*,k) \leq \sum_{j=1}^{N} \alpha_j p_j(S_j,k),$$

where the inequality is strict if and only if there exist some j and j' such that $\tilde{x}_j < k < \tilde{x}_{j'}$ with a strictly positive probability. Suppose that all securities have payoffs such that all individual options with an exercise price k will be optimally exercised simultaneously. Then a positively weighted portfolio of options on individual securities with an exercise price k will be worth just as much as an option on a portfolio of securities with the same exercise price and using the same weights. Suppose on the other hand that some options on individual securities will not be optimally exercised simultaneously. A portfolio of options, unlike an option on a portfolio of securities, gives its holder an "option" to exercise different options individually. Thus a positively weighted portfolio of call options is strictly more valuable than a call option on a portfolio of securities with the same weights.

6.8. The holder of a European put option has the right to sell its underlying security at the exercise price on the maturity date. Let $P_j(S_j, k)$ be the time–0 price of a European put written on security j with exercise price k and maturity date 1, when the current price of security j is S_j. The time–1 payoff of this put option is

$$\begin{cases} k - \tilde{x}_j > 0 & \text{if } \tilde{x}_j < k, \\ 0 & \text{if } \tilde{x}_j \geq k. \end{cases}$$

The price of a European put can be computed from the prices of its underlying security and its European call counterpart through a relation called *put–call parity*.

We claim that $P_j(S_j, k) = k/(1+r_f) - S_j + p_j(S_j, k)$. To see this, consider the following strategy: lend $k/(1+r_f)$ at the riskless rate, short sell one share of security j, and buy one share of the European call with exercise price k. The time–1 payoff of this strategy is

$$\begin{cases} k - \tilde{x}_j & \text{if } \tilde{x}_j < k, \\ 0 & \text{if } k \leq \tilde{x}_j, \end{cases}$$

which is clearly the payoff of a European put with exercise price k. To rule out arbitrage opportunities, two packages of financial assets having the same payoffs must sell for the same price. Hence our assertion follows. It is easy to see that $P_j(S_j, k)$ is an increasing function of k. Also, using arguments similar to those of Section 6.5, we can get

$$P_j(S_j, k) \geq \max[k/(1+r_f) - S_j, 0].$$

You will be asked to verify these two relations in Exercise 6.3.

Given put–call parity and the facts that the price of a call is a decreasing function of its exercise price and that the price of a put is an increasing function of its exercise price, we have

$$\frac{\partial p_j(S_j, k)}{\partial k} \geq -\frac{1}{1+r_f} \quad \text{and} \quad \frac{\partial P_j(S_j, k)}{\partial k} \leq \frac{1}{1+r_f},$$

when $p_j(S_j, k)$ and $P_j(S_j, k)$ are differentiable functions of k. Moreover, $P_j(S_j, k)$ is also a convex function of k by put–call parity.

6.9. In the previous two sections, we have seen that a call price is a decreasing and convex function of its exercise price. We will show in this section that, fixing a return distribution on the underlying asset, the price of a call option is an increasing and convex function of the price of its underlying asset. Readers are cautioned to note that this relation is a comparative statics result and is not an arbitrage relation, since different stock prices can not prevail contemporaneously.

Consider $p_j(S_j, k)$. Assume the distribution of $\tilde{x}_j/S_j$ is invariant with respect to changes in S_j. For example, if we increase S_j to S_j', then $\tilde{x}_j$ changes to $S_j'\tilde{x}_j/S_j$. We claim first that $p_j(S_j, k)$ is increasing in S_j and is strictly so if the probability that $\tilde{x}_j > k$ is strictly positive. By the assumption that return distribution is held fixed, we have

$$p_j(S_j', k) = \frac{S_j'}{S_j} p_j(S_j, kS_j/S_j') \tag{6.9.1}$$

$$\geq \frac{S_j'}{S_j} p_j(S_j, k) \geq p_j(S_j, k). \tag{6.9.2}$$

Relation (6.9.1) follows since, given that the return distribution is invariant to changes in S_j, the time–1 payoff of a call on security–j with an exercise price k when security–j's price is S_j' is equivalent to the time–1 payoff of S_j'/S_j shares of call options with an exercise price kS_j/S_j' when security j's price is S_j. Note that (6.9.1) amounts to saying that the function $p_j(S_j, k)$ is ***homogeneous of degree one*** in S_j and k. The first inequality of (6.9.2) follows since the call option price is a decreasing function of its exercise price. Finally, the second inequality of (6.9.2) follows because $S_j'/S_j > 1$ by assumption and $p_j(S_j, k) \geq 0$. The second inequality of (6.9.2) is strict if there is a strictly positive probability that $\tilde{x}_j > k$, as then $p_j(S_j, k) > 0$.

Next we want to show that $p_j(S_j, k)$ is a convex function of S_j. Let

$$\hat{S}_j \equiv \alpha S_j + (1-\alpha)S_j' \quad \text{where } \alpha \in (0,1).$$

To prove convexity, we must show that

$$\alpha p_j(S_j, k) + (1-\alpha)p_j(S_j', k) \geq p_j(\hat{S}_j, k).$$

From the fact that $p_j(S_j, k)$ is a convex function of k for all possible S_j,

$$\gamma p_j(1, k_1) + (1-\gamma)p_j(1, k_2) \geq p_j(1, k_3) \quad \forall \gamma \in (0,1), \tag{6.9.3}$$

where $k_3 \equiv \gamma k_1 + (1-\gamma)k_2$. Now take $\gamma \equiv \alpha S_j/\hat{S}_j$, $k_1 \equiv k/S_j$, and $k_2 \equiv k/S_j'$. Multiplying both sides of (6.9.3) by $\hat{S}_j$ and recalling from (6.9.1) that $p_j(S_j, k)$ is homogeneous of degree one in S_j and k,

$$\alpha p_j(S_j, S_j k_1) + (1-\alpha)p_j(S_j', S_j' k_2) \geq p_j(\hat{S}_j, \hat{S}_j k_3).$$

Using the definitions of $\hat{S}_j$, γ, k_1, k_2, and k_3, this inequality can be written as

$$\alpha p_j(S_j, k) + (1-\alpha)p_j(S_j', k) \geq p_j(\hat{S}_j, k). \tag{6.9.4}$$

The weak inequality of (6.9.4) will be strict if that of (6.9.3) is strict. From Section 6.6, we know that (6.9.3) is a strict inequality if there exists a strictly positive probability that $\tilde{x}_j$ lies between k_1 and k_2 or equivalently between k/S_j and k/S_j'.

Figure 6.9.1 illustrates the general shape that a call option price should have as a function of its underlying security price and its exercise price, while holding constant the return distribution of its underlying asset. Note that we have used (6.5.1) in Figure 6.9.1.

6.10. In this section, a pricing formula for a European call is derived under conditions on individuals' preferences and on the joint

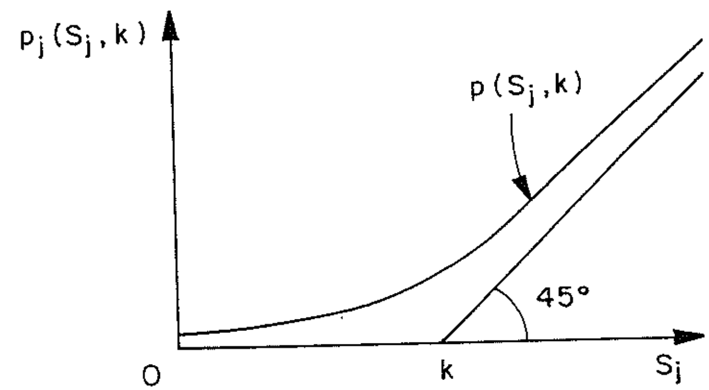

Figure 6.9.1: Call Option Price as a Function of Its Underlying Stock Price

distribution of the time–1 aggregate consumption and the payoffs of the option's underlying asset.

Consider a two–period securities markets economy. Individuals' utility functions are time–additive extended power functions with identical cautiousness as in (6.3.1). Moreover, assume that $\sum_{i=1}^{I} A_i = 0$. Recall that in a securities markets economy, an individual is endowed with time–0 consumption and shares of traded securities. From Sections 5.14 and 5.15, we know that the equilibrium allocation will be Pareto optimal, since optimal sharing rules are linear. Thus, a representative agent can be constructed with power utility functions:

$$u_0(z_0) + u_1(z_1) = \frac{1}{1-B}z_0^{1-B} + \rho\frac{1}{1-B}z_1^{1-B},$$

where ρ is the time preference parameter. Then (6.2.3) implies that

$$p_j(S_j, k) = \rho E\left[\max[\tilde{x}_j - k, 0]\left(\frac{\tilde{C}}{C_0}\right)^{-B}\right]. \tag{6.10.1}$$

We will further assume that $\tilde{x}_j$ and $\tilde{C}$ are bivariate lognormally distributed, that is, $\ln \tilde{x}_j$ and $\ln \tilde{C}$ are bivariate normally distributed with means $(\hat{\mu}_j, \hat{\mu}_c)$ and variance–covariance matrix:

$$\begin{pmatrix} \sigma_j^2 & \kappa\sigma_j\hat{\sigma}_c \\ \kappa\sigma_j\hat{\sigma}_c & \hat{\sigma}_c^2 \end{pmatrix},$$

where κ is the correlation coefficient of $\ln \tilde{x}_j$ and $\ln \tilde{C}$. This assumption implies that $\ln(\tilde{x}_j/S_j)$ and $\ln \rho(\tilde{C}/C_0)^{-B}$ are bivariate normally distributed with means

$$(\mu_j, \mu_c) \equiv (\hat{\mu}_j - \ln S_j,\ -B\hat{\mu}_c + \ln \rho + B \ln C_0)$$

and variance–covariance matrix

$$\begin{pmatrix} \sigma_j^2 & \kappa\sigma_j\sigma_c \\ \kappa\sigma_j\sigma_c & \sigma_c^2 \end{pmatrix} \equiv \begin{pmatrix} \sigma_j^2 & -B\kappa\sigma_j\hat{\sigma}_c \\ -B\kappa\sigma_j\hat{\sigma}_c & B^2\hat{\sigma}_c^2 \end{pmatrix}.$$

Given the above distributional assumption, (6.10.1) can be written as

$$p_j(S_j, k) = S_j \int_{-\infty}^{+\infty} \int_{\ln(k/S_j)}^{+\infty} (e^z - k/S_j) e^y f(z, y)\, dz\, dy, \qquad (6.10.2)$$

where $f(z, y)$ is the joint density function for $\tilde{z} \equiv \ln(\tilde{x}_j/S_j)$ and $\tilde{y} \equiv \ln \rho(\tilde{C}/C_0)^{-B}$. Relation (6.10.2) may be rewritten as the difference between two integrals

$$\begin{aligned} p_j(S_j, k) = S_j &\int_{-\infty}^{+\infty} \int_{\ln(k/S_j)}^{+\infty} e^{z+y} f(z, y)\, dz\, dy \\ - k &\int_{-\infty}^{+\infty} \int_{\ln(k/S_j)}^{+\infty} e^{y} f(z, y)\, dz\, dy. \end{aligned} \qquad (6.10.3)$$

In the next section the two integrals are evaluated and the following relations are obtained:

$$\int_{-\infty}^{\infty} \int_{a}^{\infty} e^y f(z, y)\, dz\, dy = (e^{\mu_c + \sigma_c^2/2})\ N\left(\frac{-a + \mu_j}{\sigma_j} + \kappa\sigma_c\right), \qquad (6.10.4)$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{a}^{\infty} e^{z+y} f(z, y)\, dz\, dy \\ &= (e^{\mu_j + \mu_c + (\sigma_j^2 + 2\kappa\sigma_j\sigma_c + \sigma_c^2)/2})\ N\left(\frac{-a + \mu_j}{\sigma_j} + \kappa\sigma_c + \sigma_j\right), \end{aligned} \qquad (6.10.5)$$

where $N(\cdot)$ is the distribution function of a standard normal random variable:

$$N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-v^2/2} dv,$$

and

$$n(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

is the standard normal density function. By setting $a = \ln(k/S_j)$, (6.10.4) may be used to evaluate the second integral on the right-hand side of (6.10.3), and (6.10.5) may be used to evaluate the first integral on the right side of (6.10.3) by setting $a = \ln(k/S_j)$.

It is easily verified that

$$E\left[\rho(\tilde{C}/C_0)^{-B}\right] = e^{\mu_c + \frac{1}{2}\sigma_c^2}, \qquad (6.10.6)$$

and that

$$E\left[\frac{\tilde{x}_j}{S_j}\rho\left(\frac{\tilde{C}}{C_0}\right)^{-B}\right] = e^{\mu_j + \mu_c + (\sigma_j^2 + 2\kappa\sigma_j\sigma_c + \sigma_c^2)/2}. \qquad (6.10.7)$$

The left-hand side of (6.10.6) is equal to $(1 + r_f)^{-1}$, since it gives the present value of one unit of time–1 consumption in all states. Hence,

$$e^{\mu_c + \frac{1}{2}\sigma_c^2} = (1 + r_f)^{-1}. \qquad (6.10.8)$$

Also, (6.2.3) implies that the left-hand side of (6.10.7) is equal to 1. Thus

$$e^{\mu_j + \mu_c + (\sigma_j^2 + 2\kappa\sigma_j\sigma_c + \sigma_c^2)/2} = 1. \qquad (6.10.9)$$

Now substituting (6.10.4), (6.10.5), (6.10.8), and (6.10.9) into (6.10.3) gives

$$p_j(S_j, k) = S_j N(Z_k + \sigma_j) - (1 + r_f)^{-1} k N(Z_k), \qquad (6.10.10)$$

where

$$Z_k \equiv \frac{\ln(S_j/k) + (\mu_j + \kappa\sigma_j\sigma_c)}{\sigma_j}. \qquad (6.10.11)$$

Relations (6.10.8) and (6.10.9) also imply that

$$\mu_j + \kappa\sigma_j\sigma_c = \ln(1+r_f) - \frac{1}{2}\sigma_j^2. \qquad (6.10.12)$$

Thus (6.10.11) can be written as

$$Z_k = \frac{\ln(S_j/k) + \ln(1+r_f)}{\sigma_j} - \frac{1}{2}\sigma_j \qquad (6.10.13)$$

Relations (6.10.10) and (6.10.13) are the well known *Black–Scholes option pricing formula* that was originally derived by an arbitrage argument in a continuous time economy under the assumption that the stock price follows a geometric Brownian Motion and the instantaneous riskless interest rate is a known constant. Here we derived this formula in a discrete time economy by making joint conditions on the distributions as well as on the individuals' preferences.

6.11. The derivations of (6.10.4) and (6.10.5) will be presented in this section, which are adapted from the appendix of Rubinstein (1976). Readers can skip this section without loss of continuity. The derivations of these relations require evaluations of other indefinite integrals over the marginal and conditional normal density functions. The first relation to be derived is

$$\int_a^\infty f(z)dz = N(\frac{-a+\mu_j}{\sigma_j}), \qquad (6.11.1)$$

where $f(z)$ is the marginal density function for $\ln(\tilde{x}_j/S_j)$. The first step in deriving relation (6.11.1) involves converting $f(z)$ into standard normal density function and exchanging the limits of integration, which is a valid procedure since the density function of a standard normal random variable is symmetric around zero:

$$\int_a^{+\infty} f(z)dz = \int_{\frac{a-\mu_j}{\sigma_j}}^{\infty} n(v)dv = \int_{-\infty}^{-\frac{a-\mu_j}{\sigma_j}} n(v)dv = N(\frac{-a+\mu_j}{\sigma_j}).$$

The second evaluation is

$$\begin{aligned}\int_a^{+\infty} e^z f(z)dz &= \int_a^{+\infty} \frac{1}{\sigma_j\sqrt{2\pi}}\exp\{\frac{-1}{2\sigma_j^2}(z-\mu_j)^2 + z\}dz \\ &=(e^{\mu_j+\sigma_j^2/2})\int_a^{+\infty}\frac{1}{\sigma_j\sqrt{2\pi}}\exp\{\frac{-1}{2\sigma_j^2}(z-(\mu_j+\sigma_j^2))^2\}dz \\ &=(e^{\mu_j+\sigma_j^2/2})N(\frac{-a+\mu_j}{\sigma_j}+\sigma_j).\end{aligned} \qquad (6.11.2)$$

The third evaluation is

$$\int_{-\infty}^{+\infty} e^y f(y|z)dy = \exp\{\mu_c + \kappa\frac{\sigma_c}{\sigma_j}(z-\mu_j) + \frac{1}{2}(1-\kappa^2)\sigma_c^2\}, \qquad (6.11.3)$$

where $f(y|z)$ is the conditional density of $\ln\rho(\tilde{C}/C_0)^{-B}$ given that $\ln(\tilde{x}_j/S_j) = z$. Note that the conditional distribution of $\ln\rho(\tilde{C}/C_0)^{-B}$ given that $\ln(\tilde{x}_j/S_j) = z$ is a normal distribution with mean $\mu_c + \kappa(\sigma_c/\sigma_j)(z-\mu_j)$ and variance $(1-\kappa^2)\sigma_c^2$. Then the evaluation of the left–hand side of (6.11.3) is equivalent to that of (6.11.2) by taking $a = -\infty$ and by appropriate parameter change.

Now we are prepared to derive (6.10.4). From (6.11.3) we can write

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_a^{+\infty} e^y f(z,y)dzdy &= \int_a^{+\infty} f(z)\int_{-\infty}^{+\infty} e^y f(y|z)dydz \\ &= \int_a^{+\infty} e^{\mu_c+\kappa\frac{\sigma_c}{\sigma_j}(z-\mu_j)+(1-\kappa^2)\sigma_c^2/2} f(z)dz \\ &=(e^{\mu_c+\sigma_c^2/2})\int_a^{+\infty}\frac{1}{\sigma_j\sqrt{2\pi}}\exp\{\frac{-1}{2\sigma_j^2}(z-(\mu_j+\kappa\sigma_j\sigma_c))^2\}dz,\end{aligned}$$

where $f(z)$ is the marginal density function for $\ln(\tilde{x}_j/S_j)$. Relation (6.10.4) follows by converting the density in the integral on the right–hand side of the third equality to a standard normal density and changing the limits of integration accordingly.

Finally, we are prepared to derive relation (6.10.5). From (6.11.3)

$$\int_{-\infty}^{+\infty}\int_{a}^{+\infty} e^{z}e^{y} f(z,y)dzdy = \int_{a}^{+\infty} e^{z} f(z)\int_{-\infty}^{+\infty} e^{y} f(y|z)dydz$$
$$= \int_{a}^{+\infty} e^{\mu_c+\kappa\frac{\sigma_c}{\sigma_j}(z-\mu_j)+(1-\kappa^2)\sigma_c^2/2} e^{z} f(z)dz$$
$$= e^{\mu_j+\mu_c+(\sigma_j^2+2\kappa\sigma_j\sigma_c+\sigma_c^2)/2}\int_{a}^{+\infty}\frac{1}{\sigma_j\sqrt{2\pi}}e^{-\frac{1}{2\sigma_j^2}(z-\mu_j-\kappa\sigma_j\sigma_c-\sigma_j^2)^2}dz.$$

Relation (6.10.5) is then obtained by converting the density in the integral on the right–hand side of the third equality into a standard normal density and changing the limits of the integration accordingly.

6.12. The option pricing formula in Section 6.10 is derived in an economy where individuals' utility functions exhibit linear risk tolerance with identical cautiousness and where individuals are only endowed with traded securities. In equilibrium, every individual holds a linear combination of the market portfolio and the riskless asset and a Pareto optimal allocation is achieved. Therefore, if a call option written on a security is introduced into the economy, no individual will demand it in equilibrium. Equivalently, if the economy is in equilibrium when a call option is introduced, as long as the option is priced according to (6.10.10) and (6.10.13), the original equilibrium will not be upset. The option is priced so that no individual will demand it in equilibrium. In this context, an option has no allocational role in equilibrium and is sometimes said to be a *redundant asset*. Note that the above discussion applies not only to options but also to any financial asset that is in zero net supply.

6.13. From (6.10.10) and (6.10.13), $p_j(S_j,k)$ depends upon the time–0 price of its underlying asset S_j, the exercise price k, the riskless interest rate r_f, and the variance of $\ln(\tilde{x}_j/S_j)$. It does not depend upon the mean of $\ln(\tilde{x}_j/S_j)$, however. You will be asked in Exercise 6.4 to verify the following comparative statics of $p_j(S_j,k)$ with respect to S_j, k, r_f, and σ_j:

$$\frac{\partial p_j(S_j,k)}{\partial k} = -(1+r_f)^{-1}N(Z_k) < 0, \qquad (6.13.1)$$

$$\frac{\partial p_j(S_j,k)}{\partial S_j} = N(Z_k+\sigma_j) > 0, \qquad (6.13.2)$$
$$\frac{\partial p_j(S_j,k)}{\partial \sigma_j} = k(1+r_f)^{-1}n(Z_k) > 0, \qquad (6.13.3)$$
$$\frac{\partial p_j(S_j,k)}{\partial r_f} = (1+r_f)^{-2}kN(Z_k) > 0. \qquad (6.13.4)$$

Relations (6.13.1) and (6.13.2) are confirmations of the general discussions in Sections 6.6 and 6.9. Note that in computing (6.13.2), we have assumed that the distribution of $\ln(\tilde{x}_j/S_j)$ is unchanged when we vary S_j. Relation (6.13.3) says that the higher the variance of $\ln(\tilde{x}_j/S_j)$, the more valuable the option is. This is so because an option holder does not have an obligation to exercise – he only has the right to do so. Hence, a higher σ_j allows a higher upside potential for an option. Finally, the higher the riskless interest rate, the lower the present value of the exercise price in the event of exercising at time 1, thus the more valuable the option is.

From (6.13.1), we can also get

$$\frac{\partial^2 p_j(S_j,k)}{\partial k^2} = ((1+r_f)\sigma_j k)^{-1}\, n(Z_k) > 0. \qquad (6.13.5)$$

That is, $p_j(S_j,k)$ is a convex function of the exercise price, which is a general property of call options proved earlier using an arbitrage argument. From Section 5.19, we can interpret the right–hand side of (6.13.5) to be the *price density* for a security that pays one unit of consumption at time 1 if and only if the payoff of security j is equal to k at that time. This price density is always strictly positive.

6.14. The option pricing formula derived above can be applied to study the pricing of risky corporate debt. Consider the economy in Section 6.10. Suppose that firm j has one share of common stock and a discount bond with face value k outstanding, with prices S_j and D_j, respectively. The discount bond matures at time 1. The total time–1 earning of firm j is $\tilde{x}_j$, which is assumed to be joint lognormally distributed with $\tilde{C}$, the time–1 aggregate consumption, with parameters as in Section 6.10. The present value of $\tilde{x}_j$ is the total value of the firm at time 0 and is denoted by V_j. Note that

$V_j = S_j + D_j$. The time–1 random payoff of the discount bond is $\min[\tilde{x}_j, k]$. When the firm is solvent at time 1, that is, when $\tilde{x}_j \geq k$, the bond holders receive the face value of the discount bond; otherwise, the firm goes bankrupt, and the bond holders take over the firm and get $\tilde{x}_j$. One way to compute D_j is to use (6.2.3):

$$D_j = \rho E\left[\min[\tilde{x}_j, k]\left(\frac{\tilde{C}}{C_0}\right)^{-B}\right]. \tag{6.14.1}$$

We can, however, use the Black–Scholes option pricing formula to compute D_j in a direct and straightforward way. Note that the time–1 payoff to the equity holders is $\max[\tilde{x}_j - k, 0]$. When the firm is solvent at time 1, the equity holders pay off the bond holders and get the residual value of the firm, which is $\tilde{x}_j - k$; otherwise, the bond holders take over the firm, and the equity holders get zero. Therefore, the equity holders are holding a European call on the total value of the firm with an exercise price k maturing at time 1. The value of the equity is

$$\begin{aligned} S_j &= V_j N(Z_k + \sigma_j) - (1+r_f)^{-1} k N(Z_k) \\ \text{where} \quad Z_k &\equiv \frac{\ln(V_j/k) + \ln(1+r_f)}{\sigma_j} - \frac{1}{2}\sigma_j. \end{aligned} \tag{6.14.2}$$

From the comparative statics of Section 6.13, we know that, *ceteris paribus*, the value of the equity decreases as the face value of the bond increases, and increases as σ_j increases, that is, as the total time–1 earning of the firm becomes more volatile. When the total value of the firm V_j is fixed, the increase in σ_j shifts value from the debt holders to the equity holders.

The value of the discount bond is

$$\begin{aligned} D_j &= V_j - S_j \\ &= V_j(1 - N(Z_k + \sigma_j)) + (1+r_f)^{-1} k N(Z_k). \end{aligned} \tag{6.14.3}$$

It increases as its face value increases and decreases as the earnings of the firm become more volatile.

We give two interpretations of a risky debt. The time–1 random payoff of the debt can be written as

$$\min[\tilde{x}_j, k] = \tilde{x}_j - \max[\tilde{x}_j - k, 0] \tag{6.14.4}$$

$$= k - \max[k - \tilde{x}_j, 0]. \tag{6.14.5}$$

Using (6.14.4), the bond holders can be viewed as holding the firm while selling a European call option on the value of the firm with an exercise price equal to the face value of the debt to the equity holders. We used this interpretation to compute (6.14.3). On the other hand, (6.14.5) implies that the bond holders are holding a riskless discount bond with a face value k, while at the same time selling a European put option on the value of the firm with an exercise price k. In the event that the firm's time–1 earnings are strictly less than k, the equity holders will *sell* the firm to the bond holders at a price k. Since $\tilde{x}_j$ is lognormally distributed and there, therefore, exists a strictly positive probability that the firm will default on the debt, the put option of (6.14.5) is not worthless, and the debt is risky. Therefore, the discount bond will sell at a price strictly less than $k/(1+r_f)$ and has a strictly positive risk premium.

6.15. Not all securities in strictly positive supply can have payoffs that are jointly lognormally distributed with time–1 aggregate consumption, since the sum of lognormally distributed random variables is not lognormally distributed. Under the conditions of Section 6.10, however, we can always use (6.10.10) and (6.10.13) to price European call options on the aggregate consumption/wealth. Recall from Section 5.19 that these option prices can, in turn, be used to price any complex securities.

Let $p_c(k)$ denote the price of a European call on aggregate consumption with an exercise price k maturing at time 1. Then

$$p_c(k) = S_c N(Z_k + \hat{\sigma}_c) - (1+r_f)^{-1}\, k\, N(Z_k), \tag{6.15.1}$$

where

$$Z_k \equiv \frac{\ln(S_c/k) + \ln(1+r_f)}{\hat{\sigma}_c} - \frac{1}{2}\hat{\sigma}_c, \tag{6.15.2}$$

$$\hat{\sigma}_c^2 = \operatorname{Var}(\ln(\tilde{C})), \tag{6.15.3}$$

and

$$S_c = \rho E\left[\tilde{C}\left(\frac{\tilde{C}}{C_0}\right)^{-B}\right] \tag{6.15.4}$$

is the present value of the time–1 aggregate consumption.

The price for one unit of consumption paid in states where the time 1 aggregate consumption is equal to k is

$$\Phi_c(k) = \frac{\partial^2 p_c(k)}{\partial k^2} dk = ((1+r_f)\hat{\sigma}_c k)^{-1}\, n(Z_k)dk. \qquad (6.15.5)$$

Since the probability that time–1 aggregate consumption will be equal to any fixed k is formally equal to zero, $\Phi_c(k)$ is equal to zero. (Mathematically, dk is treated as zero.) The *pricing density* $\phi_c(k) \equiv \Phi_c(k)/dk$, is strictly positive, however.

We can do a comparative statics analysis of $\phi_c(k)$. The elasticity of $\phi_c(k)$ with respect to an increase in the value of S_c is analyzed holding the distribution of $(\tilde{C}/S_c)$ constant thereby implying a proportional change in $\tilde{C}$. The resulting increase in probabilities of "high" levels of $\tilde{C}$ and decrease in probability of "low" levels of $\tilde{C}$ increase and decrease, respectively, their elementary claim prices. This elasticity is

$$\frac{\partial \ln \phi_c(k)}{\partial \ln S_c} = -\frac{Z_k}{\hat{\sigma}_c}. \qquad (6.15.6)$$

Note that Z_k is a strictly decreasing function of k and

$$\lim_{k\to 0} Z_k = \infty, \qquad \lim_{k\to\infty} Z_k = -\infty.$$

Therefore, (6.15.6) is positive for high k and negative for low k.

The elasticity of $\phi_c(k)$ with respect to the standard deviation, $\hat{\sigma}_c$, is

$$\frac{\partial \ln \phi_c(k)}{\partial \ln \hat{\sigma}_c} = (Z_k + \hat{\sigma}_c)Z_k - 1 \qquad (6.15.7)$$

This elasticity will be positive for very large and very small k and will be negative for k near $E(\tilde{C})$, because an increase in variance increases the probability of extreme observations relative to the probability of central observations.

The elasticity of the elementary claim price with respect to the riskless bond price, $(1+r_f)^{-1}$, is

$$\frac{\partial \ln \phi_c(k)}{\partial \ln (1+r_f)^{-1}} = 1 + \frac{Z_k}{\hat{\sigma}_c} \qquad (6.15.8)$$

Thus, an increase in the riskless bond price lowers the prices of claims that pay off when the level of $\tilde{C}$ is high and raise the prices of claims that pay off when the level of $\tilde{C}$ is low. However, since $(1+r_f)^{-1} = \int_0^{+\infty} \phi_c(k)dk$, an increase in $(1+r_f)^{-1}$ must increase the average elementary claim price. From (6.10.6) and (6.10.8), a change in the bond price may be associated with either a change in the expected growth rate of aggregate consumption and/or a change in relative risk aversion. Both these possibilities would provide an intuitive explanation for the resulting impact on elementary claim prices

6.16. We analyzed the comparative statics of $\phi_c(k)$ with respect to the parameters of the economy in the previous section. We can also extract some information on the structure of $\phi_c(k)$ for different levels of k. The elasticity of $\phi_c(k)$ with respect to the level of consumption on which it is contingent is

$$\frac{\partial \ln \phi_c(k)}{\partial \ln k} = \frac{Z_k}{\hat{\sigma}_c} - 1. \qquad (6.16.1)$$

For levels of k far below $E(\tilde{C})$, the elasticity will be positive because the probability density for $\tilde{C}$ increases as k increases. For k well above $E(\tilde{C})$, the elasticity will be negative due to the combined effects of the decreasing probability density of the level of consumption on which the elementary claim is contingent and of decreasing marginal utility of consumption.

6.17. Previous analyses demonstrated that lognormally distributed aggregate time–1 consumption and constant relative risk aversion utility functions for the representative agent are sufficient conditions for the Black–Scholes option pricing formula to price European options on aggregate consumption correctly. Given that time–1 aggregate consumption is lognormally distributed, a constant relative risk aversion utility function for time–1 consumption for the representative agent turns out to be also necessary for the Black–Scholes formula to price European options correctly. Recall that the pricing density of an elementary claim on aggregate consumption divided by the probability density of occurrence of that level of aggregate consumption is the marginal rate of substitution between present consumption and future consumption for the representative agent. The elasticity of this ratio with respect to aggregate consumption is constant if and only if the representative agent's utility function for time–1 consumption exhibits constant relative risk aversion. The probability density of a given level of aggregate consumption k, given S_c, is

$$\pi_c(k) = \frac{(2\pi\hat{\sigma}_c^2)^{-1/2}}{k} \exp\left\{\frac{1}{2\hat{\sigma}_c^2}[\ln(\frac{k}{S_c}) - (\mu - \hat{\sigma}_c^2)/2]^2\right\} \qquad (6.17.1)$$

Therefore, the price density of an elementary claim on aggregate consumption divided by the probability density of the occurrence of that level of aggregate consumption is

$$\frac{\phi_c(k)}{\pi_c(k)} = \frac{\exp\left[\frac{\mu - \ln(1+r_f)}{\hat{\sigma}_c^2}\ln(S_c/k) + \frac{(\mu - \ln(1+r_f))(\mu + \ln(1+r_f) - \hat{\sigma}_c^2)}{2\hat{\sigma}_c^2}\right]}{(1+r_f)}. \qquad (6.17.2)$$

The elasticity of (6.17.2) with respect to k is

$$\frac{\partial \ln(\phi_c(k)/\pi_c(k))}{\partial \ln k} = -\frac{\mu - r_f}{\hat{\sigma}_c^2} < 0, \qquad (6.17.3)$$

which is a constant. Exercise 6.5 will ask the reader to show that the negative of the above elasticity is the coefficient of relative risk aversion for the representative agent's utility function for time 1 consumption. Thus, using the Black–Scholes formula to price options on aggregate consumption is implicitly assuming that individuals' utility functions for time–1 consumption aggregate to a constant relative risk aversion utility function.

6.18. In this section we will use (6.15.5) and the pricing relation of (5.19.3) to derive the values of assets with time–1 payoffs that are jointly lognormally distributed with time–1 aggregate consumption. It will be shown that the values of such assets are appropriately determined by using a version of the CAPM.

We will use the setup and notation of Sections 6.10 and 6.15. Let $\tilde{x}_j$ be jointly lognormally distributed with time–1 aggregate consumption with parameters specified in Section 6.10. From relation

(5.19.3), we know that

$$S_j = \int_0^{+\infty} \phi_c(k) E[\tilde{x}_j|\tilde{C} = k]dk. \qquad (6.18.1)$$

Substituting (6.15.5) into (6.18.1) gives

$$S_j = \frac{1}{(1+r_f)\hat{\sigma}_c} \int_0^{+\infty} k^{-1} n(Z_k) E[\tilde{x}_j|\tilde{C} = k]dk. \qquad (6.18.2)$$

The distribution of $\ln x_j$ conditional upon $\ln \tilde{C}$ is a normal distribution with mean

$$\mu_j - \frac{1}{2}\sigma_j^2 + \ln S_j + \beta_{jc}(\ln \tilde{C} - \ln S_c - \bar{\mu}_c + \frac{1}{2}\sigma_j^2),$$

and variance

$$\sigma_j^2 - \beta_{jc}\sigma_{jc},$$

where we recall that $\mu_j = E[\ln(\tilde{x}_j/S_j)]$, where $\sigma_{jc} \equiv \mathrm{Cov}(\ln \tilde{x}_j, \ln \tilde{C})$, $\beta_{jc} \equiv \sigma_{jc}/(\sigma_j\hat{\sigma}_c)$, $\bar{\mu}_c \equiv E[\ln(\tilde{C}/S_c)]$, and S_c is defined in (6.15.4).

Note that if $\tilde{x}$ is lognormally distributed, with $E[\ln \tilde{x}] = \mu_x$ and $\mathrm{Var}(\ln \tilde{x}) = \sigma_x^2$, then $E[\tilde{x}] = \exp\{\mu_x + \sigma_x^2/2\}$. It follows that

$$E[\tilde{x}_j|\tilde{C} = k] = \exp\left\{\ln S_j + \mu_j + \beta_{jc}\left(\ln(k/S_c) - \bar{\mu}_c + \sigma_j^2/2\right) - \beta_{jc}\sigma_{jc}/2\right\}. \qquad (6.18.3)$$

Now substituting (6.18.3) into (6.18.2) and integrating gives

$$S_j = \exp\{-(\ln(1+r_f) + \beta_{jc}(\bar{\mu}_c - \ln(1+r_f)))\} E[\tilde{x}_j]. \qquad (6.18.4)$$

This implies that

$$\ln(E[\tilde{x}_j]/S_j) = \ln(1+r_f) + \beta_{jc}(\bar{\mu}_c - \ln(1+r_f)). \qquad (6.18.5)$$

As time–1 aggregate consumption is equal to time–1 aggregate wealth, $\bar{\mu}_c = E[\ln(1+\tilde{r}_m)]$ and

$$\beta_{jm} \equiv \beta_{jc} = \frac{\mathrm{Cov}(\ln(1+\tilde{r}_j), \ln(1+\tilde{r}_m))}{\sigma(\ln(1+\tilde{r}_j))\sigma(\ln(1+\tilde{r}_m))}, \qquad (6.18.6)$$

where $\tilde{r}_m$ is the rate of return on the market portfolio and $\tilde{r}_j$ is the rate of return on security j. Thus (6.18.5) can be written as

$$\ln(1+E[\tilde{r}_j]) = \ln(1+r_f)+\bar{\beta}_{jm}(E[\ln(1+\tilde{r}_m)]-\ln(1+r_f)). \quad (6.18.7)$$

That is, the log of the expected rate of return on security j satisfies a CAPM–like relationship with a beta defined in (6.18.6). Once $\bar{\beta}_{jm}$ is known, the present value of security j can simply be computed by discounting its expected time–1 payoff at the exponential of the continuously compounded *risk adjusted discount rate*, which is (6.18.7).

Exercises

6.1. Derive a pricing relation similar to (6.3.3) when individuals have log utility functions.

6.2. Show that the price of a European call option is a decreasing function of its exercise price, and find the conditions under which it is a strictly decreasing function of its exercise price.

6.3. Let $P_j(S_j,k)$ be the price of a European put on security j with an exercise price k maturing at time 1. Show that P_j is an increasing function of k and that, when the probability that $\tilde{x}_j < k$ lies strictly between 0 and 1,

$$P_j(S_j,k) > \max[k/(1+r_f) - S_j, 0].$$

6.4. Verify relations (6.13.1) to (6.13.5).

6.5. Prove that (6.17.3) is the relative risk aversion of the representative agent's utility function for time–1 consumption.

6.6. We derived the Black–Scholes option pricing formula by assuming that the representative agent's utility functions for time–0 as well as for time–1 consumption are of constant relative risk aversion. Derive a similar pricing formula by assuming only that the representative agent's utility function for time–1 consumption exhibits constant relative risk aversion.

Remarks. The discussion on skewness preferences follows Kraus and Litzenberger (1976, 1983). Discussions in Sections 6.5–6.9 are taken from Merton (1973), in which readers can also find a host of related subjects. Our derivation of the Black–Scholes option pricing formula is from Rubinstein (1976). Unlike the original Black–Scholes derivation, Rubinstein's derivation uses equilibrium arguments. Black and Scholes (1973) use arbitrage arguments and derive their formula in a continuous time economy. Merton (1973) formalizes and extends the Black–Scholes results. Cox, Ross, and Rubinstein (1979) also use arbitrage arguments to derive an option pricing formula in a discrete time economy by assuming that risky stock prices follow a binomial random walk. This subject is covered in Chapter 8. They show that their formula converges to the Black–Scholes formula when the trading intervals shrink to zero and when appropriate limits of their price system are taken. The application of the Black–Scholes option pricing formula to the pricing of corporate risky debt is adapted from Merton (1974). The discussions in Sections 6.15–6.18 are taken from Breeden and Litzenberger (1978). An option pricing formula under the assumptions that individuals have negative exponential utility functions and that asset returns are multivariate normally distributed is derived by Brennan (1979). For a review of the recent developments in option pricing theory and its applications see Cox and Huang (1987).